Question 4

Let $d \geq 0$ be an integer and V be a $(2d+1)$-dimensional complex linear space with a basis

$$\{v_1, v_2, \cdots, v_{2d+1}\}.$$

For an integer j $\left(0 \leq j \leq \frac{d}{2}\right)$, write U_j for the subspace generated by

$$v_{2j+1}, v_{2j+3}, \cdots, v_{2d-2j+1}.$$

Define a linear transformation $f : V \to V$ by

$$f(v_i) = \frac{(i-1)(2d+2-i)}{2} v_{i-1} + \frac{1}{2} v_{i+1}, \quad 1 \leq i \leq 2d+1.$$

Here we put $v_0 = v_{2d+2} = 0$.

1. Show that eigenvalues of f are $-d, -d+1, \ldots, d$.
2. Write W for the sum of eigenspaces of f of eigenvalues $-d+2k$ $(0 \leq k \leq d)$. Find the dimension of $W \cap U_0$.
3. For any integer j $\left(1 \leq j \leq \frac{d}{2}\right)$, find the dimension of $W \cap U_j$.